

# EXTREMAL PROPERTIES OF FINITE ULTRAMETRIC SPACES AND INVARIANTS OF THEIR ISOMETRIES

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**ABSTRACT.** Some extremal properties of finite ultrametric spaces and related properties of representing trees are described. We also describe conditions under which the isomorphism of representing trees is equivalent to the isometricity of corresponding finite ultrametric spaces.

## 1. INTRODUCTION

In 2001 at the Workshop on General Algebra [1] the attention of experts on the theory of lattices was paid to the following problem of I. M. Gelfand: Using graph theory describe up to isometry all finite ultrametric spaces. An appropriate representation of ultrametric spaces  $X$  by monotone rooted trees  $T_X$  was proposed in [2]. The representation from [2] can be considered in some sense as a solution of above mentioned problem. The question naturally arises about applications of this representation. One such application is the structural characteristic of finite ultrametric spaces for which the Gomory-Hu inequality becomes an equality, see [3]. The ultrametric spaces for which its representing trees are strictly binary were described in [4]. Our paper is also a contribution to these lines of studies.

The first section of the paper contains the main definition and the required technical results. In the second section we describe some extremal and structural properties of finite metric spaces which have the strictly  $n$ -ary representing trees and the representing trees with injective internal labeling. The main results here are Theorem 2.4, Theorem 2.8 and Corollary 2.13. It is clear that the representing tree and the range of distance function are invariant under isometries of finite ultrametric spaces. Theorem 3.7 and Theorem 3.9 of the third section of the paper contain a description of finite ultrametric spaces for which the converse also holds: If spaces have the same representing trees and the same range of distance function, then these spaces are isometric.

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Recall some definitions from the theory of metric spaces and the graph theory.

**Definition 1.1.** An *ultrametric* on a set  $X$  is a function  $d: X \times X \rightarrow \mathbb{R}^+$ ,  $\mathbb{R}^+ = [0, \infty)$ , such that for all  $x, y, z \in X$ :

- (i)  $d(x, y) = d(y, x)$ ,
- (ii)  $(d(x, y) = 0) \Leftrightarrow (x = y)$ ,
- (iii)  $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ .

Inequality (iii) is often called the *strong triangle inequality*. The *spectrum* of an ultrametric space  $(X, d)$  is the set

$$\text{Sp}(X) = \{d(x, y) : x, y \in X\}.$$

The quantity

$$\text{diam } X = \sup\{d(x, y) : x, y \in X\}.$$

is the diameter of the space  $(X, d)$ . Recall that a *graph* is a pair  $(V, E)$  consisting of a nonempty set  $V$  and a (probably empty) set  $E$  whose elements are unordered pairs of different points from  $V$ . For a graph  $G = (V, E)$ , the sets  $V = V(G)$  and  $E = E(G)$  are called *the set of vertices (or nodes)* and *the set of edges*, respectively. If  $\{x, y\} \in E(G)$ , then the vertices  $x$  and  $y$  are adjacent. A graph  $G = (V, E)$  together with a function  $w: E \rightarrow \mathbb{R}^+$  is called a *weighted graph*, and  $w$  is called a *weight* or a *weighting function*. The weighted graphs  $G$  will be denoted as  $(G, w)$ . A graph is *complete* if  $\{x, y\} \in E(G)$  for all distinct  $x, y \in V(G)$ . Recall that a *path* is a nonempty graph  $P = (V, E)$  whose vertices can be numbered so that

$$V = \{x_0, x_1, \dots, x_k\}, \quad E = \{\{x_0, x_1\}, \dots, \{x_{k-1}, x_k\}\}.$$

A *Hamiltonian path* in a graph  $G$  is a path that visits each vertex of  $G$  exactly once. A finite graph  $C$  is a *cycle* if  $|V(C)| \geq 3$  and there exists an enumeration  $(v_1, v_2, \dots, v_n)$  of its vertices such that

$$(\{v_i, v_j\} \in E(C)) \Leftrightarrow (|i - j| = 1 \quad \text{or} \quad |i - j| = n - 1).$$

A graph  $H$  is a *subgraph* of a graph  $G$  if

$$V(H) \subseteq V(G) \quad \text{and} \quad E(H) \subseteq E(G).$$

A cycle  $C$  is *Hamiltonian* in a graph  $G$  if  $C$  is a subgraph of  $G$  such that  $V(C) = V(G)$ . A connected graph without cycles is called a *tree*. A tree  $T$  may have a distinguished vertex called the *root*; in this case  $T$  is called a *rooted tree*. Generally we follow terminology used in [5].

**Definition 1.2.** Let  $k \geq 2$ . A nonempty graph  $G$  is called *complete  $k$ -partite* if its vertices can be divided into  $k$  disjoint nonempty sets  $X_1, \dots, X_k$  so that there are no edges joining the vertices of the same

set  $X_i$  and any two vertices from different  $X_i, X_j$ ,  $1 \leq i, j \leq k$  are adjacent. In this case we write  $G = G[X_1, \dots, X_k]$ .

We shall say that  $G$  is a *complete multipartite graph* if there exists  $k \geq 2$  such that  $G$  is complete  $k$ -partite, cf. [6].

**Definition 1.3** ([7]). Let  $(X, d)$  be a finite ultrametric space. Define the graph  $G_X^d$  as follows  $V(G_X^d) = X$  and

$$(\{u, v\} \in E(G_X^d)) \Leftrightarrow (d(u, v) = \text{diam } X).$$

We call  $G_X^d$  the *diametrical graph* of  $X$ .

**Theorem 1.4** ([7]). Let  $(X, d)$  be a finite ultrametric space,  $|X| \geq 2$ . Then  $G_X^d$  is complete multipartite.

With every finite ultrametric space  $(X, d)$ , we can associate a labeled rooted tree  $T_X$  by the following rule (see [3]).

If  $X = \{x\}$  is a one-point set, then  $T_X$  is the rooted tree consisting of one node  $X$  with the label 0. Note that for the rooted trees consisting only of one node, we consider that this node is the root as well as a leaf.

Let  $|X| \geq 2$ . According to Theorem 1.4 we have  $G_X^d = G_X^d[X_1, \dots, X_k]$ ,  $k \geq 2$ . In this case the root of the tree  $T_X$  is labeled by  $\text{diam } X$  and, moreover,  $T_X$  has  $k$  nodes  $X_1, \dots, X_k$  of the first level with the labels

$$(1.1) \quad l(X_i) = \text{diam } X_i,$$

$i = 1, \dots, k$ . The nodes of the first level with the label 0 are leaves, and those indicated by strictly positive labels are internal nodes of the tree  $T_X$ . If the first level has no internal nodes, then the tree  $T_X$  is constructed. Otherwise, by repeating the above-described procedure with  $X_i$  instead of  $X$ , we obtain the nodes of the second level, etc. Since  $|X|$  is finite, and the cardinal numbers  $|Y|$ ,  $Y \subseteq X$ , decrease strictly at the motion along any path starting from the root, all vertices on some level will be leaves, and the construction of  $T_X$  is completed.

The above-constructed labeled tree  $T_X$  is called the *representing tree* of the space  $(X, d)$ . We say that a function  $l: V(T_X) \rightarrow \mathbb{R}^+$  satisfying (1.1) is the labeling of  $T_X$ . Note that the range of the labeling  $l: V(T_X) \rightarrow \mathbb{R}^+$  is  $\text{Sp}(X)$ . The restriction of the labeling  $l: V(T_X) \rightarrow \mathbb{R}^+$  on the set of internal node of  $T_X$  is called the internal labeling of  $T_X$ . Note that the correspondence between the trees and the ultrametric spaces is well known [2, 8–11].

In 1961 E. C. Gomory and T. C. Hu [12], for arbitrary finite ultrametric space  $X$ , proved the inequality

$$|\text{Sp}(X)| \leq |X|.$$

Denote by  $\mathfrak{U}$  the class of finite ultrametric spaces  $X$  with  $|\mathrm{Sp}(X)| = |X|$ . In [3] two descriptions of  $X \in \mathfrak{U}$  were obtained (see Theorem 1.7 below).

**Definition 1.5.** Let  $(X, d)$  be an ultrametric space with  $|X| \geq 2$  and the spectrum  $\mathrm{Sp}(X)$  and let  $r \in \mathrm{Sp}(X)$  be nonzero. Define by  $G_{r,X}$  a graph for which  $V(G_{r,X}) = X$  and

$$(\{u, v\} \in E(G_{r,X})) \Leftrightarrow (d(u, v) = r).$$

For  $r = \mathrm{diam} X$  it is clear that  $G_{r,X}$  is the diametrical graph of  $X$ .

**Definition 1.6.** Let  $G = (V, E)$  be a nonempty graph, and let  $V_0$  be the set (possibly empty) of all isolated vertices of  $G$ . Denote by  $G'$  the subgraph of the graph  $G$ , induced by the set  $V \setminus V_0$ .

Recall that a rooted tree is *strictly  $n$ -ary* if every its internal node has exactly  $n$  children. In the case  $n = 2$  such tree is called *strictly binary*.

**Theorem 1.7** ([3]). *Let  $(X, d)$  be a finite ultrametric space with  $|X| \geq 2$ . The following conditions are equivalent.*

- (i)  $(X, d) \in \mathfrak{U}$ .
- (ii)  $G'_{r,X}$  is complete bipartite for every nonzero  $r \in \mathrm{Sp}(X)$ .
- (iii)  $T_X$  is strictly binary and the internal labeling of  $T_X$  is injective.

Another criterium of  $X \in \mathfrak{U}$  in terms of weighted Hamiltonian cycles and weighted Hamiltonian paths was proved in [4].

Let  $(X, d)$  be a finite ultrametric space and let  $Y$  be a nonempty subspace of  $X$ . In the next proposition we identify  $Y$  with the complete weighted graph  $(G_Y, w)$  such that  $V(G_Y) = Y$  and

$$(1.2) \quad \forall x, y \in Y: \quad w(\{x, y\}) = d(x, y) \text{ if } x \neq y.$$

**Proposition 1.8** ([4]). *Let  $(X, d)$  be a finite nonempty ultrametric space. The following conditions are equivalent.*

- (i)  $T_X$  is strictly binary.
- (ii) If  $Y \subseteq X$  and  $|Y| \geq 3$ , then there exists a Hamiltonian cycle in  $(G_Y, w)$  with exactly two edges of maximal weight.
- (iii) There is no equilateral triangle in  $(X, d)$ .

In the next section of the paper we characterize the finite ultrametric spaces having the strictly  $n$ -ary representing trees and the finite ultrametric spaces having the representing trees with injective internal labeling.

2. INJECTIVE INTERNAL LABELING AND STRICTLY  $n$ -ARY TREES

Let  $T = T(r)$  be a labeled rooted tree with the root  $r$ . We denote by  $\overline{L}_T$  the set of the leaves of  $T$ , and, for every node  $v$  of  $T$ , by  $l(v)$  the label of  $v$ , and by  $T_v = T(v)$  the induced rooted subtree of  $T$  whose nodes defined by the rule

$$(2.1) \quad (u \in V(T_v)) \Leftrightarrow (u = v \text{ or } u \text{ is a successor of } v).$$

If  $(X, d)$  be a finite ultrametric space and  $T = T_X$ , where  $T_X$  is the representing tree of  $X$ , then for every node  $v \in V(T)$  there are  $x_1, \dots, x_k \in X$  such that  $\overline{L}_{T_v} = \{\{x_1\}, \dots, \{x_k\}\}$ . Thus  $\overline{L}_{T_v}$  is a set of one-point subsets of  $X$ . In what follows we will use the notation  $L_{T_v}$  for the set  $\{x_1, \dots, x_k\}$ .

The following lemma was proved in [3] for the spaces  $X \in \mathfrak{U}$  but its proof is also true for arbitrary finite ultrametric spaces.

**Lemma 2.1.** *Let  $(X, d)$  be a finite ultrametric space and let  $u_1 = \{x_1\}$  and  $u_2 = \{x_2\}$  be two different leaves of the tree  $T_X$ . If  $(u_1, v_1, \dots, v_n, u_2)$  is the path joining the leaves  $u_1$  and  $u_2$  in  $T_X$ , then*

$$(2.2) \quad d(x_1, x_2) = \max_{1 \leq i \leq n} l(v_i).$$

Recall that the union  $G_1 \cup G_2$  of graphs  $G_1$  and  $G_2$  is the graph with the vertex set  $V(G_1) \cup V(G_2)$  and the edge set  $E(G_1) \cup E(G_2)$ . If  $V(G_1) \cap V(G_2) = \emptyset$ , then the union is *disjoint*.

**Lemma 2.2.** *Let  $(X, d)$  be a finite ultrametric space, let  $T_X$  be its representing tree and let  $r \in \text{Sp}(X) \setminus \{0\}$ . Then the graph  $G'_{r,X}$  is the disjoint union of  $p$  complete multipartite graphs  $G_{r,X}^1, \dots, G_{r,X}^p$  where  $p$  is the number of nodes  $x_1, \dots, x_p$  labeled by  $r$ . Moreover for every  $i \in \{1, \dots, p\}$  the graph  $G_{r,X}^i$  is complete  $k$ -partite,*

$$(2.3) \quad G_{r,X}^i = G_{r,X}^i[L_{T_{s_1}}, \dots, L_{T_{s_k}}],$$

where  $s_1, \dots, s_k$  are the direct successors of  $x_i$ .

*Proof.* Suppose first that there is a single node  $x_1$  labeled by  $r$ . Let  $s_1, \dots, s_k$  be the direct successors of  $x_1$ . Consider the subtrees  $T_{s_1}, \dots, T_{s_k}$  of  $T_X$  with roots  $s_1, \dots, s_k$ . Let

$$x, y \in \bigcup_{j=1}^k L_{T_{s_j}}.$$

According to Lemma 2.1 and to the construction of the representing trees the equality  $d(x, y) = r$  holds if and only if  $x$  and  $y$  belong to the different sets  $L_{T_{s_1}}, \dots, L_{T_{s_k}}$ . Using Definitions 1.5 and 1.2 we see that  $G'_{r,X}$  is complete  $k$ -partite with parts  $L_{T_{s_1}}, \dots, L_{T_{s_k}}$ .

Let  $x_1, \dots, x_p$  be the internal nodes labeled by  $r$ . According to the construction of the representing trees there are no  $i, j \in \{1, \dots, p\}$  such that  $x_i$  is a successor of  $x_j$ . This means that  $L_{T_{x_i}} \cap L_{T_{x_j}} = \emptyset$  for all  $i, j \in \{1, \dots, p\}$ ,  $i \neq j$ . Arguing as above we see that every node  $x_i$  generates a complete multipartite graph  $G_{r,X}^i$ , with  $V(G_{r,X}^i) = L_{T_{x_i}}$  such that  $V(G_{r,X}^i) \cap V(G_{r,X}^j) = \emptyset$  for all distinct  $i, j \in \{1, \dots, p\}$ .  $\square$

Let  $(X, d)$  be a metric space. Recall that a *ball* with a radius  $r \geq 0$  and a center  $t \in X$  is the set  $B_r(t) = \{x \in X : d(x, t) \leq r\}$ . By  $\mathbf{B}_X$  we denote the set of all balls of the space  $(X, d)$ . Every one-point subset of  $X$  belongs to  $\mathbf{B}_X$ , we will say that this is a *singular* ball in  $X$ .

The proof of the next lemma can be found in [13] but we reproduce it here for the convenience of the reader.

**Lemma 2.3** ([13]). *Let  $(X, d)$  be a finite ultrametric space with representing tree  $T_X$ ,  $|X| \geq 2$ . Then the following statements hold.*

- (i)  $L_{T_v} \in \mathbf{B}_X$  holds for every node  $v \in V(T_X)$ .
- (ii) For every  $B \in \mathbf{B}_X$  there exists the node  $v$  such that  $L_{T_v} = B$ .

*Proof.* (i) In the case where  $v$  is a leaf of  $T_X$  statement (i) is evident. Let  $v$  be an internal node of  $T_X$  and let  $\{t\} \in L_{T_v}$ . Consider the ball

$$B_{l(v)}(t) = \{x \in X : d(x, t) \leq l(v)\}.$$

Let  $\{t_1\} \in L_{T_v}$ ,  $t_1 \neq t$ . Since the path joining  $\{t\}$  and  $\{t_1\}$  lies in the tree  $T_v$ , we have  $d(t, t_1) \leq l(v)$  (see Lemma 2.1). The inclusion  $L_{T_v} \subseteq B_{l(v)}(t)$  is proved. Conversely, suppose there exists  $t_0 \in B_{l(v)}(t)$  such that  $\{t_0\} \notin L_{T_v}$ . Consider the path  $(\{t_0\}, v_1, \dots, v_n, \{t\})$ . It is clear that  $\max_{1 \leq i \leq n} l(v_i) > l(v)$ , i.e.,  $d(t_0, t) > l(v)$ . We have a contradiction.

(ii) In the case  $|B| = 1$  statement (ii) is evident. Let  $t \in X$  and let  $B \in \mathbf{B}_X$  such that  $|B| \geq 2$ . Let  $x, y \in B$  with  $d(x, y) = \text{diam } B$ . Consider the path  $(\{x\}, v_1, \dots, v_n, \{y\})$  in the tree  $T_X$ . According to Lemma 2.1 we have  $d(x, y) = \max_{1 \leq i \leq n} l(v_i)$ . Let  $i$  be an index such that  $d(x, y) = l(v_i)$ . Put  $v = v_i$ . The proof of the equality  $L_{T_v} = B$  is analogous to the proof of (i).  $\square$

**Theorem 2.4.** *Let  $(X, d)$  be a finite nonempty ultrametric space. The following conditions are equivalent.*

- (i) *The diameters of different nonsingular balls are different.*
- (ii) *The internal labeling of  $T_X$  is injective.*
- (iii)  *$G'_{r,X}$  is a complete multipartite graph for every  $r \in \text{Sp}(X) \setminus \{0\}$ .*
- (iv) *The equality*

$$|\mathbf{B}_X| = |X| + |\text{Sp}(X)| - 1$$

holds.

*Proof.* The theorem is trivial if  $|X| = 1$ . Suppose that  $|X| \geq 2$ . Lemma 2.3 and the definition of the representing trees imply that (i) and (ii) are equivalent. The implication (ii) $\Rightarrow$ (iii) follows from Lemma 2.2.

Let us prove (iii) $\Rightarrow$ (ii). Suppose that  $G'_{r,X}$  is a complete multipartite graph for every  $r \in \text{Sp}(X) \setminus \{0\}$ . Consider the case where there exist exactly two different internal nodes  $u$  and  $v$  of  $T_X$  with  $l(u) = l(v)$ . According to properties of the representing trees  $u$  and  $v$  are not incident. Let  $u_1, \dots, u_m$  and  $v_1, \dots, v_n$  be the direct successors of  $u$  and  $v$  respectively. Arguing as in proof of Lemma 2.2 we see that  $G'_{l,X} = U \cup W$ , where  $l = l(u) = l(v)$  and

$$U = U[L_{T_{u_1}}, \dots, L_{T_{u_m}}], \quad W = W[L_{T_{v_1}}, \dots, L_{T_{v_n}}] \quad \text{and} \quad V(U) \cap V(W) = \emptyset.$$

We have a contradiction since the disjoint union of two complete multipartite graphs is disconnected. The case where the number of different internal nodes having equal labels is more than two is analogous.

The equivalence (ii) $\Leftrightarrow$ (iv) follows from Lemma 2.3 and from the equality

$$|Y| = |f(Y)|,$$

which holds for every injective mapping  $f$  defined on a set  $Y$ . The proof is completed.  $\square$

*Remark 2.5.* The inequality

$$(2.4) \quad |\text{Sp}(X)| \leq |\mathbf{B}_X| - |X| + 1$$

holds for every finite nonempty ultrametric space  $X$ . Indeed, it follows from Lemma 2.3 that

$$|\mathbf{B}_X| = |V(T_X)|.$$

Each node of  $T_X$  is either an internal node or a leaf. Since  $|\text{Sp}(X)| - 1$  is no more than the number of internal nodes and since the number of leaves is  $|X|$ , inequality (2.4) holds. Thus, by Theorem 2.4, the internal labeling of  $T_X$  is injective if and only if inequality (2.4) becomes an equality.

**Lemma 2.6.** *Let  $X$  be a finite nonempty ultrametric space. Suppose that there is a natural number  $n \geq 2$  such that the equality*

$$(2.5) \quad (n-1)|\mathbf{B}_Y| + 1 = n|Y|$$

*holds for every ball  $Y \in \mathbf{B}_X$ . Then the following statements are equivalent for every nonsingular ball  $Y \in \mathbf{B}_X$ .*

- (i) *The equality  $|Y| = n$  holds.*

(ii) *All children of the node  $Y$  are leaves of  $T_X$ .*

*Proof.* Let  $|Y| = n$  hold. Then from (2.5) it follows that

$$(n-1)|\mathbf{B}_Y| = n^2 - 1.$$

Since  $n-1 \neq 0$ , the last equality implies that  $|\mathbf{B}_Y| = n+1$ . The set  $Y$  and the one-point subsets of  $Y$  are elements of  $\mathbf{B}_Y$ . Hence, the equalities  $|\mathbf{B}_Y| = n+1$  and  $|Y| = n$  give us either  $B = Y$  or  $|B| = 1$ . Statement (ii) follows from (i).

The converse also holds. Indeed, suppose that  $Y \in \mathbf{B}_X$  is nonsingular and all children of  $Y$  are leaves. Then we have the equality

$$|\mathbf{B}_Y| = |Y| + 1.$$

This equality and (2.5) imply the equation

$$(n-1)(|Y| + 1) + 1 = n|Y|$$

which has the unique solution  $|Y| = n$ . □

*Remark 2.7.* If  $|Y| = 1$ , then we have also  $|\mathbf{B}_Y| = 1$ . Consequently, equality (2.5) holds for every ultrametric space  $X$  and every positive integer number  $n$  if  $Y$  a singular ball.

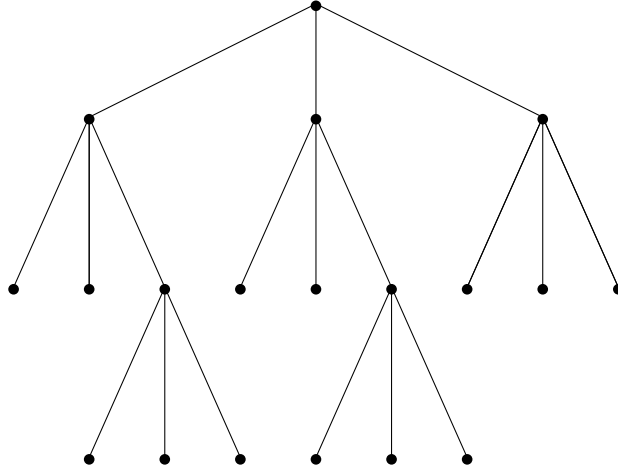


FIGURE 1. Example of strictly 3-ary tree.

Let  $(X, d)$  be an ultrametric space. Recall that balls  $B_1, \dots, B_k$  in  $(X, d)$  is equidistant if there is  $r > 0$  such that  $d(x_i, x_j) = r$  holds whenever  $x_i \in B_i$  and  $x_j \in B_j$  and  $1 \leq i < j \leq k$ . Every two disjoint balls  $B_1, B_2$  in  $(X, d)$  are equidistant.

**Theorem 2.8.** *Let  $(X, d)$  be a finite nonempty ultrametric space and let  $n \geq 2$  be integer. The following conditions are equivalent.*



- (i)  $T_X$  is strictly  $n$ -ary.
- (ii) For every nonzero  $r \in \text{Sp}(X)$  the graph  $G'_{r,X}$  is the union of  $p$  complete  $n$ -partite graphs, where  $p$  is a number of all internal nodes of  $T_X$  labeled by  $r$ .
- (iii) For every nonsingular ball  $B \in \mathbf{B}_X$  there are the equidistant balls  $B_1, \dots, B_n \in \mathbf{B}_X$  such that  $B = \bigcup_{j=1}^n B_j$  and  $\text{diam } B_j < \text{diam } B$  for every  $j \in \{1, \dots, n\}$ .
- (iv) The equality (2.5) holds for every ball  $Y \in \mathbf{B}_X$ .

*Proof.* (i) $\Rightarrow$ (ii) This implication follows directly from Lemma 2.2.

(ii) $\Rightarrow$ (i) Let condition (ii) hold and let  $x_1, \dots, x_p$  be the internal nodes of  $T_X$  labeled by  $r \in \text{Sp}(X) \setminus \{0\}$ . Consider the trees  $T_{x_i}$  with the roots  $x_i$ ,  $i = 1, \dots, p$ . It follows from the definition of the representing trees that among the nodes  $x_1, \dots, x_p$  there are no adjacent. Consequently the sets  $L_{T_{x_1}}, \dots, L_{T_{x_p}}$  are disjoint. Hence every complete  $n$ -partite graph from the union mentioned in condition (ii) is generated by an internal node  $x_i$  labeled by  $r$ . Using the fact that this graph is  $n$ -partite we see that the node  $x_i$  has exactly  $n$  direct successors for every  $i \in \{1, \dots, p\}$ .

The construction of the representing trees and Lemma 2.3 yield the equivalence (i) $\Leftrightarrow$ (iii).

(i) $\Rightarrow$ (iv) Suppose (i) holds. Let us denote by  $N_Y$  the number of the internal nodes of  $T_Y$  for arbitrary nonsingular  $Y \in \mathbf{B}_Y$ . It follows directly from the definition of the strictly  $n$ -ary rooted trees and Lemma 2.3 that  $T_Y$  is strictly  $n$ -ary. Hence we have  $nN_Y + 1 = |V(T_Y)|$ . Lemma 2.3 implies  $|V(T_Y)| = |\mathbf{B}_Y|$  and  $N_Y = |\mathbf{B}_Y| - |Y|$ . Consequently,

$$n(|\mathbf{B}_Y| - |Y|) + 1 = |\mathbf{B}_Y|$$

holds. The last equality and (2.5) are equivalent.

(iv) $\Rightarrow$ (i) Let (iv) hold. Equality (2.5) implies that

$$(2.6) \quad |Y| = |\mathbf{B}_Y| + \frac{1}{n} - \frac{|\mathbf{B}_Y|}{n}$$

for every nonsingular ball  $Y \in \mathbf{B}_X$ . Since

$$(2.7) \quad |\mathbf{B}_Y| \geq |Y| + 1,$$

from (2.6) we obtain

$$|\mathbf{B}_Y| \geq \left( |\mathbf{B}_Y| + \frac{1}{n} - \frac{|\mathbf{B}_Y|}{n} \right) + 1$$

that gives the inequality

$$|\mathbf{B}_Y| \geq n + 1.$$

Now using (2.6) we see that

$$|Y| = \left(1 - \frac{1}{n}\right) |\mathbf{B}_Y| + \frac{1}{n} \geq \frac{(n-1)(n+1)}{n} + \frac{1}{n} = n.$$

Thus every nonsingular ball  $Y \in \mathbf{B}_X$  contains at least  $n$  distinct points.

Now we can prove (i) by induction on  $|X|$ .

It was proved that  $|Y| \geq n$  holds for every nonsingular ball  $Y \in \mathbf{B}_X$ . Consequently we have  $|X| \geq n$ . If  $|X| = n$ , then statement (i) follows from Lemma 2.6. Suppose (i) does not hold if  $|X| = m$  but (i) holds if  $n \leq |X| < m$ . The space  $(X, d)$  contains a nonsingular ball  $Y \in \mathbf{B}_X$  such that all successors of the node  $Y$  are leaves of  $T_X$ . Define a set  $X^*$  as

$$X^* = (X \setminus Y) \cup \{y^*\}$$

where  $y^*$  is an arbitrary point of  $Y$ . Since  $|Y| \geq 2$  we have the inequality

$$|X^*| < |X|.$$

Hence to complete the proof of (i) it suffices, by the induction hypothesis, to show that

$$(2.8) \quad (n-1)|\mathbf{B}_W| + 1 = n|W|$$

holds for every  $W \in \mathbf{B}_{X^*}$ . Let  $W \in \mathbf{B}_{X^*}$ . Equality (2.8) is trivial if  $y^* \notin W$ . Let  $y^* \in W$ . Then the set

$$\tilde{W} = W \cup Y$$

is a nonsingular ball in  $(X, d)$ . It is easy to prove that

$$|\tilde{W}| = |W| + |Y| - 1 \quad \text{and} \quad |\mathbf{B}_{\tilde{W}}| = |\mathbf{B}_W| + |Y|.$$

These equalities and Lemma 2.6 give us

$$(2.9) \quad |\tilde{W}| = |W| + n - 1 \quad \text{and} \quad |\mathbf{B}_{\tilde{W}}| = |\mathbf{B}_W| + n.$$

Using (2.9) and equality (2.5) with  $Y = \tilde{W}$  we obtain

$$(n-1)(|\mathbf{B}_W| + n) + 1 = n(|W| + n - 1).$$

Equality (2.8) follows.  $\square$

Let  $T$  be a rooted tree and let  $v$  be a node of  $T$ . Denote by  $\delta^+(v)$  the out-degree of  $v$ , i.e.,  $\delta^+(v)$  is the number of children of  $v$ , and write

$$\Delta^+(T) = \max_{v \in V(T)} \delta^+(v),$$

i.e.,  $\Delta^+(T)$  is the maximum out-degree of  $V(T)$ . It is clear that  $v \in V(T)$  is a leaf of  $T$  if and only if  $\delta^+(v) = 0$ . Moreover,  $T$  is strictly  $n$ -ary if and only if the equality

$$\delta^+(v) = n$$

holds for every internal node  $v$  of  $T$ . Let us denote by  $I(T)$  the set of all internal nodes of  $T$ .

**Lemma 2.9.** *The inequality*

$$(2.10) \quad |V(T)| \leq \Delta^+(T)|I(T)| + 1$$

*holds for every rooted tree  $T$ . If  $|V(T)| \geq 2$ , then this inequality becomes the equality if and only if  $T$  is strictly  $n$ -ary with  $n = \Delta^+(T)$ .*

*Proof.* Let  $T$  be a rooted tree. It is clear that

$$(2.11) \quad |E(T)| = \sum_{v \in I(T)} \delta^+(v).$$

Since  $|V(T)| = |E(T)| + 1$  holds (see, for example, [6, Corollary 1.5.3]) and we have

$$(2.12) \quad \sum_{v \in I(T)} \delta^+(v) \leq \Delta^+(T)|I(T)|,$$

inequality (2.10) follows.

It is easy to see that inequalities (2.10) and (2.12) become equalities simultaneously. Since  $\delta^+(v) \leq \Delta^+(T)$  holds for every  $v \in V(T)$ , inequality (2.12) becomes an equality if and only if we have

$$\delta^+(v) = \Delta^+(T)$$

for every  $v \in I(T)$ . The last condition means that  $T$  is strictly  $n$ -ary with  $n = \Delta^+(T)$ .  $\square$

**Corollary 2.10.** *The inequality*

$$(2.13) \quad |\mathbf{B}_X| \geq \frac{\Delta^+(T_X)|X| - 1}{\Delta^+(T_X) - 1}$$

*holds for every finite nonempty ultrametric space  $(X, d)$ . This inequality becomes an equality if and only if  $T_X$  is a strictly  $n$ -ary rooted tree with  $n = \Delta^+(T_X)$ .*

*Proof.* If  $|X| = 1$ , then we have  $|\mathbf{B}_X| = 1$  and  $\Delta^+(T_X) = 0$ . Thus

$$\frac{\Delta^+(T_X)|X| - 1}{\Delta^+(T_X) - 1} = \frac{0 - 1}{0 - 1} = 1 = |\mathbf{B}_X|.$$

Suppose  $|X| \geq 2$  holds. It follows from Lemma 2.3 that

$$|I(T_X)| = |\mathbf{B}_X| - |X| \quad \text{and} \quad |V(T_X)| = |\mathbf{B}_X|.$$

Thus (2.13) is an equivalent form of (2.10) for  $T = T_X$ .  $\square$

*Remark 2.11.* We have  $\Delta^+(T_X) - 1 \neq 0$  in inequality (2.14), see Lemma 3.4 in the third section of the paper.

Using Corollary 2.10 and Remark 2.5 we obtain

**Proposition 2.12.** *Let  $(X, d)$  be a finite ultrametric space with  $|X| \geq 2$ . Then the inequality*

$$(2.14) \quad 2|\mathbf{B}_X| \geq |\mathrm{Sp}(X)| + \frac{2\Delta^+(T_X)|X| - \Delta^+(T_X) - |X|}{\Delta^+(T_X) - 1}$$

*holds. This inequality becomes an equality if and only if  $T_X$  is a strictly  $n$ -ary rooted tree with  $n = \Delta^+(T_X)$  and with the injective internal labeling.*

*Proof.* Note that the right side of (2.14) is the sum of the right sides of inequalities (2.4) and (2.13). The proposition follows from Remark 2.5 and Corollary 2.10.  $\square$

**Corollary 2.13.** *Let  $(X, d)$  be a finite ultrametric space with  $|X| \geq 2$  and let  $n = \Delta^+(T_X)$ . The following conditions are equivalent.*

- (i)  $T_X$  is a strictly  $n$ -ary tree with injective internal labeling.
- (ii)  $G'_{r,X}$  is complete  $n$ -partite graph for every  $r \in \mathrm{Sp}(X)$ .
- (iii) The equality

$$(2.15) \quad 2|\mathbf{B}_X| = |\mathrm{Sp}(X)| + \frac{2\Delta^+(T_X)|X| - \Delta^+(T_X) - |X|}{\Delta^+(T_X) - 1}$$

*holds.*

Theorem 1.7 and Corollary 2.13 show, in particular, that equality  $|X| = |\mathrm{Sp}(X)|$  holds if and only if we have (2.15) and  $\Delta^+(T_X) = 2$ .

In the following proposition we suppose that the weighted graph  $(G_Y, w)$  is defined as in Proposition 1.8.

**Proposition 2.14.** *Let  $(X, d)$  be a finite ultrametric space with  $|X| \geq 3$  and let  $n \geq 2$  be a natural number. If  $T_X$  is strictly  $n$ -ary, then for every nonsingular ball  $Y \in \mathbf{B}_X$ , the graph  $(G_Y, w)$  contains a Hamiltonian cycle with exactly  $n$  edges of maximal weight.*

*Proof.* Let  $T_X$  be strictly  $n$ -ary and let  $Y \in \mathbf{B}_X$  with  $|Y| > 1$ . According to Lemma 2.3 there exists an internal node  $x_0$  of  $T_X$  such that  $Y = L_{T_{x_0}}$ . Since  $T_X$  is strictly  $n$ -ary,  $x_0$  has  $n$  direct successors  $x_1, \dots, x_n$ . Consider the subtrees  $T_{x_1}, \dots, T_{x_n}$  with the roots  $x_1, \dots, x_n$ . By Lemma 2.1 for any  $x, y \in Y$  the equality  $d(x, y) = l(x_0)$  holds if  $x$  and  $y$  belong to different sets  $L_{T_{x_i}}, L_{T_{x_j}}$ ,  $1 \leq i, j \leq n$ , and we have  $d(x, y) < l(x_0)$  if  $x$  and  $y$  belong to the same one. It is easy to see that the cycle

$$C(x_{11}, \dots, x_{1k_1}, x_{21}, \dots, x_{2k_2}, \dots, x_{n1}, \dots, x_{nk_n})$$

is Hamiltonian for the weighted graph  $(G_Y, w)$  with exactly  $n$  edges of maximal weight  $l(x_0) = \text{diam } Y$  where

$$\{x_{11}, \dots, x_{1k_1}\} = L_{T_{x_1}}, \dots, \{x_{n1}, \dots, x_{nk_n}\} = L_{T_{x_n}}.$$

□

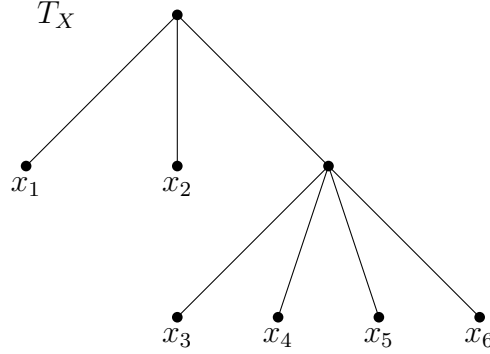


FIGURE 2. The space  $X$  for which  $T_X$  is not strictly 4-ary.

*Remark 2.15.* The inversion of Proposition 2.14 is not true. The existence for every nonsingular ball  $Y \in \mathbf{B}_X$  of a Hamiltonian cycle with exactly  $n$  edges of maximal weight in  $(G_Y, w)$ , does not guarantee that the rooted tree  $T_X$  is strictly  $n$ -ary.

*Example 2.16.* Let  $(X, d)$  be an ultrametric space with  $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$  and with  $T_X$  depicted at Figure 2. There are only two nonsingular balls  $B_1 = X$  and  $B_2 = \{x_3, x_4, x_5, x_6\}$  in  $(X, d)$ . It is clear that both Hamiltonian cycles  $C_1 = (x_1, x_3, x_2, x_4, x_5, x_6)$  and  $C_2 = (x_3, x_4, x_5, x_6)$  of the graphs  $(G_{B_1}, w)$  and  $(G_{B_2}, w)$ , respectively, have exactly 4 edges of maximal weight.

### 3. FROM ISOMORPHIC TREES TO ISOMETRIC SPACES

Recall that two metric spaces  $(X, d)$  and  $(Y, \rho)$  are isometric if there is a bijection  $f: X \rightarrow Y$  such that the equality

$$d(x, y) = \rho(f(x), f(y))$$

holds for all  $x, y \in X$ . It is clear that we have

$$\text{Sp}(X) = \text{Sp}(Y)$$

for isometric  $(X, d)$  and  $(Y, \rho)$ .

**Definition 3.1.** The rooted trees  $T_1 = T_1(r_1)$  and  $T_2 = T_2(r_2)$  are isomorphic if there is a bijection  $f: V(T_1) \rightarrow V(T_2)$  such that

$$(3.1) \quad f(r_1) = r_2 \quad \text{and} \quad (\{u, v\} \in E(T_1)) \Leftrightarrow (\{f(u), f(v)\} \in E(T_2))$$

holds for all distinct  $u, v \in V(T)$ . We will write  $T_1 \simeq T_2$  if  $T_1$  and  $T_2$  are isomorphic.

It is easy to prove that any two isometric finite ultrametric spaces  $(X, d)$  and  $(Y, \rho)$  have isomorphic representing trees  $T_X$  and  $T_Y$ . The converse does not hold. The spaces  $(X, d)$  and  $(Y, \rho)$  can fail to be isometric even if  $T_X$  is isomorphic to  $T_Y$  and  $\text{Sp}(X) = \text{Sp}(Y)$ .

Let us denote by  $\mathcal{TSI}$  (tree-spectrum isometric) the class of all finite ultrametric spaces  $(X, d)$  which satisfy the following condition:

If  $(Y, \rho)$  is a finite ultrametric space such that  $T_X \simeq T_Y$  and  $\text{Sp}(X) = \text{Sp}(Y)$ , then  $(X, d)$  and  $(Y, \rho)$  are isometric.

In this section we describe characteristic structural properties of representing trees  $T_X$  for spaces  $(X, d)$  belonging to some subclasses of  $\mathcal{TSI}$ .

**Definition 3.2.** Let  $T_i = T_i(r_i, l_i)$  be labeled rooted trees with the roots  $r_i$  and the labelings  $l_i: V(T_i) \rightarrow \mathbb{R}^+$ ,  $i = 1, 2$ . A bijection  $f: V(T_1) \rightarrow V(T_2)$  is an isomorphism of  $T_1(r_1, l_1)$  and  $T_2(r_2, l_2)$  if (3.1) holds for all distinct  $u, v \in V(T_1)$  and, moreover, we have

$$(3.2) \quad l_2(f(v)) = l_1(v)$$

for every  $v \in V(T_1)$ . The labeled rooted trees  $T_1(r_1, l_1)$  and  $T_2(r_2, l_2)$  are isomorphic if there is an isomorphism  $f: V(T_1) \rightarrow V(T_2)$ .

The following lemma is a reformulation of Theorem 2.6 from [14].

**Lemma 3.3.** *Let  $(X, d)$  and  $(Y, \rho)$  be nonempty finite ultrametric spaces. Then the labeled rooted trees  $T_X$  and  $T_Y$  with the labelings defined as in (1.1) are isomorphic if and only if  $(X, d)$  and  $(Y, \rho)$  are isometric.*

*Proof.* Let  $\Psi: X \rightarrow Y$  be an isometry. It follows from Lemma 2.3 and the definition of the representing trees that

$$V(T_X) = \mathbf{B}_X \quad \text{and} \quad V(T_Y) = \mathbf{B}_Y.$$

Since  $\Psi: X \rightarrow Y$  is an isometry, the set

$$\{\Psi(x): x \in A\}, \quad A \subseteq X$$

is a ball in  $(Y, \rho)$  if and only if  $A$  is a ball in  $(X, d)$ . Consequently we can define a bijection  $f: \mathbf{B}_X \rightarrow \mathbf{B}_Y$  by the rule

$$(3.3) \quad f(B) = \{\Psi(x): x \in B\}, \quad B \in \mathbf{B}_X.$$

A ball  $B_1 \in \mathbf{B}_X$  is a direct successor of a ball  $B_2 \in \mathbf{B}_X$  if and only if

$$(3.4) \quad (B_1 \subseteq B \subseteq B_2) \Rightarrow ((B_1 = B) \vee (B_2 = B))$$

holds for every  $B \in \mathbf{B}_X$ . Since  $\Psi$  is an isometry, (3.4) is an equivalent of

$$(f(B_1) \subseteq f(B) \subseteq f(B_2)) \Rightarrow ((f(B_1) = f(B)) \vee (f(B_2) = f(B))).$$

Thus (3.1) holds for all  $B_1, B_2 \in \mathbf{B}_X$  if  $f$  is defined by (3.3) and  $T_1 = T_X$  and  $T_2 = T_Y$ . Moreover, for  $T_1 = T_X$  and  $T_2 = T_Y$ , equality (3.2) is an equivalent for

$$\text{diam}(\Psi(B)) = \text{diam}(B),$$

that holds because  $\Psi$  is an isometry. Moreover it is clear that  $f(X) = Y$ . Hence the labeled rooted trees are isomorphic if  $(X, d)$  and  $(Y, \rho)$  are isometric.

The converse is also valid. To see it, note that the restriction of an isomorphism

$$f: V(T_X) \rightarrow V(T_Y)$$

of labeled rooted trees  $T_X$  and  $T_Y$  on the set  $\overline{L}_T$  of leaves of  $T_X$  gives us the bijection

$$(3.5) \quad \overline{L}_{T_X} \ni v \mapsto f(v) \in \overline{L}_{T_Y}.$$

Since  $v$  and  $f(v)$  are some one-point subsets of  $X$  and of  $Y$  respectively, there is the unique bijection  $\Psi: X \rightarrow Y$  such that

$$(\Psi(x) = y) \Leftrightarrow (f(\{x\}) = \{y\})$$

holds for all  $x \in X$  and  $y \in Y$ . Lemma 2.1 implies that the bijection  $\Psi$  is an isometry because  $f$  is an isomorphism of the labeled rooted trees  $T_X$  and  $T_Y$ . □

**Lemma 3.4.** *Let  $T = T(r, l_T)$  be a labeled rooted tree with the root  $r$  and the labeling  $l_T: V(T) \rightarrow \mathbb{R}^+$ . Then the following two conditions are equivalent.*

(i) *For every  $u \in V(T)$  we have  $\delta^+(u) \neq 1$  and*

$$((\delta^+(u) = 0) \Leftrightarrow (l_T(u) = 0))$$

*and, in addition, the inequality*

$$(3.6) \quad l_T(v) < l_T(u)$$

*holds whenever  $v$  is a direct successor of  $u$ .*

(ii) *There is a finite ultrametric space  $(X, d)$  such that  $T_X$  and  $T$  are isomorphic in the sense of Definition 3.2.*

*Proof.* (i)  $\Rightarrow$  (ii) Let us denote by  $X$  the set of the leaves of  $T$ . For every pair  $x, y \in X$  denote by  $P_{x,y}$  the subset of  $V(T)$  consisting of all nodes  $w$  for which  $x$  and  $y$  are successors of  $w$  and write

$$(3.7) \quad d(x, y) = \min_{w \in P_{x,y}} l_T(w).$$

Using condition (i) we can prove that the function  $d: X \times X \rightarrow \mathbb{R}^+$  is an ultrametric on  $X$ . Now (3.7), the definition of the representing trees and Lemma 2.1 imply that  $T_X$  and  $T(r, l_T)$  are isomorphic in the sense of Definition 3.2.

(ii)  $\Rightarrow$  (i) If  $(X, d)$  is a finite ultrametric space, then condition (i) evidently holds for  $T = T_X$ . Moreover, if we have two labeled rooted trees which are isomorphic in the sense of Definition 3.2 and one of them satisfies condition (ii), then another tree also satisfies (ii). The implication (ii)  $\Rightarrow$  (i) holds.  $\square$

Lemma 3.4 implies, in particular, the following corollaries.

**Corollary 3.5.** *Let  $T = T(r)$  be a rooted tree. Then the following conditions are equivalent.*

- (i) *For every  $u \in V(T)$  we have  $\delta^+(u) \neq 1$ .*
- (ii) *There is a finite ultrametric space  $(X, d)$  such that  $T_X$  and  $T$  are isomorphic in the sense of Definition 3.1.*

**Corollary 3.6.** *For every finite rooted tree  $T = T(r)$  there is a finite ultrametric space  $(X, d)$  such that  $T$  is isomorphic to the subtree  $\tilde{T}_X$  of the representing tree  $T_X$  induced by the set  $I(T_X)$  of all internal nodes of  $T_X$ .*

*Proof.* Let  $V(T) = \{v_1, \dots, v_n\}$  and let  $U = \{u_1, \dots, u_n\}$  and  $W = \{w_1, \dots, w_n\}$  be some sets such that

$$U \cap W = U \cap V(T) = W \cap V(T) = \emptyset.$$

We define a graph  $G$  by the rule

$$V(G) = U \cup W \cup V(T)$$

and

$$E(G) = E(T) \cup \{\{v_1, u_1\}, \dots, \{v_n, u_n\}\} \cup \{\{v_1, w_1\}, \dots, \{v_n, w_n\}\}.$$

It is easy to prove that  $G$  is a tree, see Figure 3. For every vertex  $x$  of the rooted tree  $G(r)$  we have the inequality  $\delta^+(x) \geq 2$ . Hence by Corollary 3.5 there is a finite ultrametric space  $(X, d)$  such that  $T_X$  and  $G(r)$  are isomorphic in the sense of Definition 3.1. Consequently  $\tilde{T}_X$  and  $\tilde{G}(r)$  are also isomorphic in the sense of Definition 3.1, where



$\tilde{G}(r)$  is the subtree of  $G(r)$  induced by the set of all internal nodes of  $G(r)$ . To complete the proof it suffices to note that  $\tilde{G}(r) = T(r)$ .  $\square$

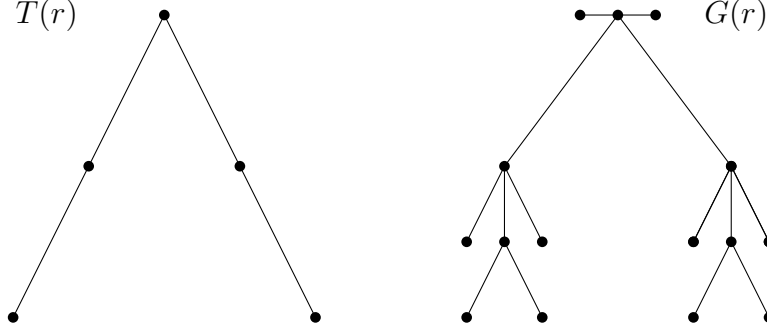


FIGURE 3. Each tree is a tree of nonsingular balls of finite ultrametric space.

Let  $T$  be a rooted tree. The *height* of  $T$  is the number of edges on the longest path between the root and a leaf of  $T$ . The height of  $T$  will be denoted by  $h(T)$ . For every node  $v$  of  $T$ , the level of  $v$  can be defined by the following inductive rule: The level of the root of  $T$  is zero and if  $v \in V(T)$  has a level  $x$ , then every direct successor of  $v$  has the level  $x + 1$ . We denote by  $\text{lev}(v)$  the level of a node  $v \in V(T)$ . Thus

$$(3.8) \quad h(T) = \max_{v \in \bar{L}_T} \text{lev}(v).$$

**Theorem 3.7.** *Let  $(X, d)$  be a finite ultrametric space with  $|X| \geq 2$ . Suppose that the representing tree  $T_X$  has an injective internal labeling. Then the following statements are equivalent.*

- (i) *The space  $(X, d)$  belongs to  $\mathcal{TSI}$ .*
- (ii) *The equality  $\text{lev}(v) = \text{lev}(u)$  implies*

$$\text{lev}(v) = \text{lev}(u) = h(T_X) - 1 \text{ and } \delta^+(u) = \delta^+(v)$$

*for all distinct internal nodes  $u, v \in V(T_X)$ .*

*Proof.* The root of  $T_X$  is the unique node with the level zero and, in addition, each node  $u$  with  $\text{lev}(u) = h(T_X)$  is a leaf of  $T_X$ . Hence if (ii) does not hold, then we have exactly one from the following two alternatives:

- (i<sub>1</sub>) *There are two distinct internal nodes  $u, v \in V(T_X)$  and  $k \in \{1, \dots, h(T_X) - 2\}$  such that*

$$(3.9) \quad \text{lev}(u) = \text{lev}(v) = k;$$

- (i<sub>2</sub>) For every  $k \in \{1, \dots, h(T_X) - 2\}$  there exists exactly one  $w \in I(T_X)$  with  $\text{lev}(w) = k$  but there exist at least two distinct nodes  $u, v \in I(T_X)$  such that

$$(3.10) \quad \text{lev}(u) = \text{lev}(v) = h(T) - 1 \text{ and } \delta^+(u) \neq \delta^+(v).$$

(i)  $\Rightarrow$  (ii) Let  $(X, d) \in \mathcal{TSI}$ . Suppose (i<sub>1</sub>) holds. Denote by  $k_0$  the smallest  $k \in \{1, \dots, h(T_X) - 2\}$  which satisfies (3.9). If for every  $w \in I(T_X)$  with  $\text{lev}(w) = k$  all children of  $w$  are leaves of  $T_X$ , then we have

$$k_0 = h(T_X) - 1.$$

The last equality contradicts the condition  $k_0 \in \{1, \dots, h(T_X) - 2\}$ . Hence there is at least one  $w \in I(T_X)$  satisfying the equality

$$\text{lev}(w) = k_0$$

and having a direct successor which is an internal node of  $T_X$ . Without loss of generality we suppose  $w \neq u$ . Define the sets  $W$  and  $U$  as

$$W = I(T_w) \text{ and } U = I(T_u),$$

see (2.1). Since  $\text{lev}(u) = \text{lev}(w)$ , the sets  $W$  and  $U$  are disjoint. Write  $n = |W|$  and  $m = |U|$ . Then we have  $n+m \geq 3$ . Let  $l: V(T_X) \rightarrow [0, \infty)$  be the labeling of  $T_X$  defined by (1.1) and let

$$S_w = \{l(t) : t \in W\} \text{ and } S_u = \{l(t) : t \in U\}.$$

Since the restriction  $l|_{I(T_X)}$  is injective, we have

$$|S_w| = n, \quad |S_u| = m \text{ and } S_w \cap S_u = \emptyset$$

and, in addition,

$$0 \notin S_w \cup S_u.$$

All elements  $l^w \in S_w$  and  $l^u \in S_u$  can be enumerated such that

$$(3.11) \quad l_1^w < l_2^w < \dots < l_n^w \text{ and } l_1^u < l_2^u < \dots < l_m^u.$$

Since  $n+m \geq 3$ , there are disjoint subsets  $S_w^*$  and  $S_u^*$  of the set  $S_w \cup S_u$  such that

$$S_u^* \cup S_w^* = S_w \cup S_u$$

and

$$|S_w^*| = n, \quad |S_u^*| = m$$

and

$$(3.12) \quad S_u \neq S_w^* \neq S_w \neq S_u^* \neq S_u.$$

Similarly to (3.11) we suppose that the elements of  $S_u^*$  and  $S_w^*$  are enumerated as

$$(3.13) \quad l_1^{*w} < l_2^{*w} < \dots < l_n^{*w} \text{ and } l_1^{*u} < l_2^{*u} < \dots < l_m^{*u},$$

$l_i^{*w} \in S_w^*$ ,  $i = 1, \dots, n$ , and  $l_j^{*u} \in S_u^*$ ,  $j = 1, \dots, m$ .

Let us define a new labeling  $l^*: V(T_X) \rightarrow \mathbb{R}^+$  by the rule

$$(3.14) \quad l^*(v) = \begin{cases} l(v), & \text{if } v \in V(T_X) \setminus (W \cup U), \\ l_i^{*w}, & \text{if } v \in W \text{ and } l(v) = l_i^w, \ i \in \{1, \dots, n\}, \\ l_j^{*u}, & \text{if } v \in U \text{ and } l(v) = l_j^u, \ j \in \{1, \dots, m\}. \end{cases}$$

Using (3.11), (3.12) and Lemma 3.4 we can prove that there is an ultrametric  $d^*$  on the set  $X^* = X$  such that the rooted trees  $T_X$  and  $T_{X^*}$  are isomorphic and

$$\text{Sp}(X) = \text{Sp}(X^*).$$

Since  $(X, d) \in \mathcal{TSI}$ , the trees  $T_X$  and  $T_{X^*}$  are also isomorphic as the labeled rooted trees (See Lemma 3.3). Let  $f: V(T_X) \rightarrow V(T_{X^*})$  be the corresponding isomorphism. Since  $f$  preserves the labeling, equality (3.14) imply that

$$f(v) = v$$

holds for every internal node  $v \in V(T_X) \setminus (W \cup U)$ . Moreover

$$\text{lev}(v) = \text{lev}(f(v))$$

holds for every  $v \in V(T)$ . Hence we have either  $f(w) = u$  or  $f(w) = w$ . If  $f(w) = u$ , then  $S_W = S_U^*$  holds, contrary to (3.12). Similarly if  $f(w) = w$ , then we obtain  $S_W = S_W^*$ , that also contradicts (3.12).

Suppose  $(i_2)$  is valid. Let  $u$  and  $v$  be two distinct internal nodes of  $T_X$  for which (3.10) holds. Consider the new labeling  $l^*: V(T_X) \rightarrow \mathbb{R}^+$ ,

$$l^*(w) = \begin{cases} l(w), & \text{if } u \neq w \neq v, \\ l(u), & \text{if } w = v, \\ l(v), & \text{if } w = u. \end{cases}$$

By Lemma 3.3 we can prove that there is an ultrametric  $d^*$  on the set  $X^* = X$  such that  $l^*$  is the labeling on the rooted tree  $T_{X^*}$  generated by the ultrametric space  $(X^*, d^*)$ . It is easy to see that  $T_X$  and  $T_{X^*}$  are isomorphic as rooted trees and we have  $\text{Sp}(X^*) = \text{Sp}(X)$ . Now  $(i)$  implies that  $(X, d)$  and  $(X^*, d^*)$  are isometric, which is impossible because there are the unique ball  $B$  in  $(X, d)$  with the diameter  $l(u)$  and the unique ball  $B^*$  in  $(X^*, d^*)$  with same diameter and such that  $|B| = \delta^+(u) \neq \delta^+(v) = |B^*|$ .

$(ii) \Rightarrow (i)$  Let  $(Y, \rho)$  be a finite ultrametric space with  $\text{Sp}(Y) = \text{Sp}(X)$  and let  $f: V(T_X) \rightarrow V(T_Y)$  be an isomorphism of the rooted trees  $T_X$  and  $T_Y$ . We first prove that the internal labeling of  $T_Y$  is injective. This is equivalent to that

$$(3.15) \quad |\mathbf{B}_Y| = |Y| + |\text{Sp}(Y)| - 1$$

holds (see Theorem 2.4). The equality  $\text{Sp}(X) = \text{Sp}(Y)$  implies

$$(3.16) \quad |\text{Sp}(X)| = |\text{Sp}(Y)|.$$

Moreover, we have

$$(3.17) \quad |\mathbf{B}_Y| = |\mathbf{B}_X| \text{ and } |X| = |Y|$$

because  $f$  is an isomorphism of the rooted trees  $T_X$  and  $T_Y$ . Since the internal labeling of  $T_X$  is injective

$$(3.18) \quad |\mathbf{B}_X| = |X| + |\text{Sp}(X)| - 1$$

holds. Now (3.15) follows from (3.16), (3.17) and (3.18).

Let  $\tilde{T}_X$  and  $\tilde{T}_Y$  be the subtrees of  $T_X$  and, respectively, of  $T_Y$  induced by  $I(T_X)$  and  $I(T_Y)$  respectively. Using (ii) we can prove that the function  $\tilde{\Psi}: V(\tilde{T}_X) \rightarrow V(\tilde{T}_Y)$  defined by the rule

$$(3.19) \quad (\tilde{\Psi}(u) = v) \Leftrightarrow (v \in \tilde{T}_Y \text{ and } \text{diam } u = \text{diam } v)$$

is an isomorphism of the labeled rooted trees  $\tilde{T}_X$  and  $\tilde{T}_Y$ . Note that rule (3.19) is correct because the internal labelings of  $T_X$  and  $T_Y$  are injective and  $\text{Sp}(X) = \text{Sp}(Y)$ . The leaves of the trees  $\tilde{T}_X$  and  $\tilde{T}_Y$  are the internal nodes of  $T_X$  at the level  $h(T_X) - 1$  and, respectively, of  $T_Y$  at the level  $h(T_Y) - 1$ . By statement (ii) the number of children is one and the same for all these nodes. Consequently  $\tilde{\Psi}$  can be extended to an isomorphism  $\Psi: V(T_X) \rightarrow V(T_Y)$  of the labeled rooted trees  $T_X$  and  $T_Y$ . By Lemma 3.3, we obtain that  $(X, d)$  and  $(Y, \rho)$  are isometric.  $\square$

*Remark 3.8.* Statement (ii) means that  $T_X$  has exactly one internal node at each level except the levels  $h(T)$  and  $h(T) - 1$  and the number of children is constant for all internal nodes at the level  $h(T) - 1$ . See Figure 4 for example of a representing tree satisfying this property.

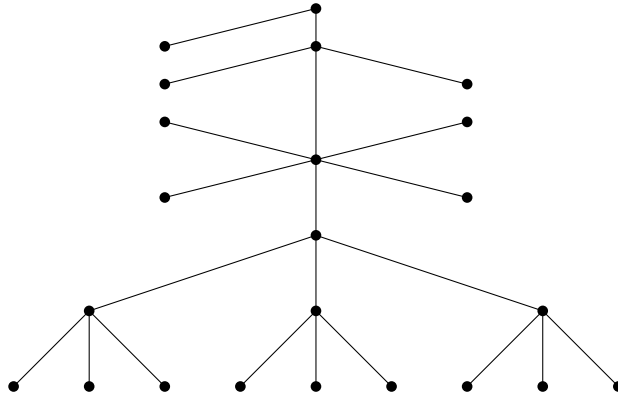


FIGURE 4

**Theorem 3.9.** *Let  $T = T(r)$  be a rooted tree with  $\delta^+(u) \geq 2$  for every internal node  $u$ . Then the following conditions are equivalent.*

- (i) *The tree  $T$  contains exactly one internal node at the levels  $1, \dots, h(T) - 2$  and at most two internal nodes at the level  $h(T) - 1$ . If  $u$  and  $v$  are different internal nodes with*

$$\text{lev}(u) = \text{lev}(v) = h(T) - 1,$$

*then  $\delta^+(u) = \delta^+(v)$  holds.*

- (ii) *For every finite ultrametric space  $(X, d)$  the statement  $T_X \simeq T$  implies that  $(X, d) \in \mathcal{TSI}$ .*

*Proof.* (i)  $\Rightarrow$  (ii). Let condition (i) hold and let  $X, Y$  be finite ultrametric spaces such that  $T_X \simeq T \simeq T_Y$  and  $\text{Sp}(X) = \text{Sp}(Y)$ . Let  $\Psi: V(T_X) \rightarrow T_Y$  be an isomorphism of the rooted trees  $T_X$  and  $T_Y$ . Suppose first there is exactly one internal node of  $T$  at the level  $h(T) - 1$ . Then it is easy to see that

$$X \ni x \mapsto \{x\} \mapsto f(\{x\}) \mapsto y \in Y,$$

is an isometry, where  $y$  is the unique point of the one-point set  $f(\{x\})$ .

Let  $u, v \in I(T_X)$  and  $\bar{u}, \bar{v} \in I(T_Y)$  and let

$$\text{lev } u = \text{lev } v = \text{lev } \bar{u} = \text{lev } \bar{v} = h(T) - 1$$

and  $u \neq v$  and  $\bar{u} \neq \bar{v}$ . Taking into consideration the structure of the tree  $T$  and the construction of the representing trees of finite ultrametric spaces it is easy to see that there are only two variants

- (i<sub>1</sub>) The nodes  $u, v, \bar{u}$  and  $\bar{v}$  are labeled by the same number in the case  $|\text{Sp}(X)| = |\text{Sp}(Y)| = h(T) + 1$ .
- (i<sub>2</sub>) The nodes  $u, v$  and  $\bar{u}, \bar{v}$  are labeled by the two smallest elements of the set  $\text{Sp}(X)$  in the case  $|\text{Sp}(X)| = |\text{Sp}(Y)| = h(T) + 2$ .

The existence of isometry in the case (i<sub>1</sub>) is almost evident. For the case (i<sub>2</sub>) without loss of generality suppose that  $l(u) = l(\bar{u})$  and  $l(v) = l(\bar{v})$ . Let  $\Psi: V(T_X) \rightarrow V(T_Y)$  be the isomorphism of  $T_X$  and  $T_Y$  such that  $\Psi(u) = \bar{u}$  and  $\Psi(v) = \bar{v}$ . Doing direct calculations we see that

$$X \ni x \mapsto \{x\} \mapsto \Psi(\{x\}) \mapsto y \in Y$$

is an isometry between  $X$  and  $Y$ , where  $y$  is the unique element of the one-point set  $\Psi(\{x\})$ .

- (ii) Suppose the implication

$$(3.20) \quad (T_X \simeq T(r)) \Rightarrow ((X, d) \in \mathcal{TSI})$$

holds for every finite ultrametric space  $(X, d)$ . Let us consider a labeling  $l_T: V(T) \rightarrow \mathbb{R}^+$  such that the restriction  $l_T|_{I(T)}$  is injective and  $l_T(u) = 0$  for every  $u \in \bar{L}_T$  and  $l_T(v) < l_T(w)$ , whenever  $v$  is a direct successor

of  $w$ . By Lemma 3.4 there is a finite ultrametric space  $(X, d)$  such that  $T_X$  and the labeled rooted tree  $T(r, l_T)$  are isomorphic. Theorem 3.7 implies that

$$\text{lev}(v) = \text{lev}(u) = h(T_X) - 1 \text{ and } \delta^+(u) = \delta^+(v)$$

for all distinct  $u, v \in I(T_X)$ . Since  $T_X$  and  $T = T(r, l_T)$  are isomorphic, the similar statement holds for  $T(r, l_T)$ . Suppose now that

$$u_1, \dots, u_p \in I_T, \quad p \geq 3$$

and

$$\text{lev}(u_1) = \dots = \text{lev}(u_p) = h(T) - 1.$$

Let  $l^j: V(T) \rightarrow \mathbb{R}^+$ ,  $j = 1, 2$ , be labelings of  $T$  such that

$$l^1(u) = l^2(u) = 0$$

for all  $u \in \overline{L}_T$  and

$$l^j(v) < l^j(w), \quad j = 1, 2$$

whenever  $v$  is a direct successor of  $w$  and that

$$(3.21) \quad l^1(u_1) \neq l^2(u_1), \quad l^2(u_k) = l^1(u_1), \quad l^1(u_k) = l^2(u_1)$$

for  $k = 2, \dots, p$ . Let  $(X_1, d_1)$  and  $(X_2, d_2)$  be finite ultrametric spaces which correspond to the labeled rooted trees  $T(r, l^1)$  and  $T(r, l^2)$  by Lemma 3.4. Then  $\text{Sp}(X_1) = \text{Sp}(X_2)$  but  $(X_1, d_1)$  and  $(X_2, d_2)$  are not isometric, contrary to (3.20).  $\square$

Theorem 3.9 gives us a sufficient condition for  $(X, d) \in \mathcal{TSI}$  which are independent of the properties of  $\text{Sp}(X)$ . Analogously to this theorem, we may also provide the membership  $(X, d) \in \mathcal{TSI}$  using only  $\text{Sp}(X)$ .

**Proposition 3.10.** *Let  $A$  be a finite subset of  $\mathbb{R}^+$  and let  $0 \in A$ . Then the following conditions are equivalent*

- (i) *The inequality  $|A| \leq 3$  holds.*
- (ii) *For every finite ultrametric space  $(X, d)$  the equality  $\text{Sp}(X) = A$  implies  $(X, d) \in \mathcal{TSI}$ .*

The proof is simple, so that we omit it here. See Figure 5 for the example of representing trees of finite ultrametric spaces  $(X, d)$  with  $|\text{Sp}(X)| = 3$ .

In the rest of this section we describe extremal properties of some subclass of finite ultrametric spaces considered in Theorem 3.7 and Theorem 3.9.

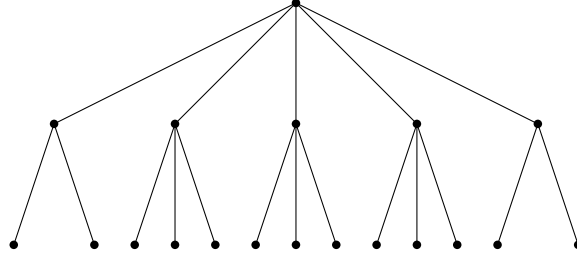


FIGURE 5

**Proposition 3.11.** *Let  $(X, d)$  be a finite nonempty ultrametric space. Then the inequality*

$$(3.22) \quad h(T_X) \leq |\text{Sp}(X)| - 1$$

*holds.*

*Proof.* By Lemma 3.3 the inequality

$$l(u) < l(v)$$

holds whenever  $u$  is a direct successor of  $v$ . Now (3.22) follows from (1.1) because

$$\text{diam}(Y) \in \text{Sp}(X)$$

for every  $Y \in \mathbf{B}_X$ . □

**Lemma 3.12.** *The inequality*

$$(3.23) \quad h(T_X) \leq |\mathbf{B}_X| - |X|$$

*holds for every finite nonempty ultrametric space  $(X, d)$ . This inequality becomes the equality if and only if*

$$h(T_X) = |\text{Sp}(X)| - 1 \quad \text{and} \quad |\text{Sp}(X)| = |\mathbf{B}_X| - |X| - 1.$$

*Proof.* Let  $(X, d)$  be a finite nonempty ultrametric space. Using Proposition 3.11 and Remark 2.5 we obtain the double inequality

$$h(T_X) + 1 \leq |\text{Sp}(X)| \leq |\mathbf{B}_X| - |X| + 1,$$

that implies the desirable result. □

**Lemma 3.13.** *Let  $(X, d)$  be a finite ultrametric space with  $|X| \geq 2$ , and let  $T_X$  have exactly one internal node at each level except the last level. Then for every  $Y \subseteq X$ ,  $|Y| \geq 2$ , the representing tree  $T_Y$  of the space  $(Y, d)$  also has exactly one internal node at each level except the last level.*

*Proof.* Since every subset of  $X$  can be obtained by consecutive deleting of a points, it is sufficient to prove this implication in the case  $Y = X \setminus \{x_i\}$ ,  $x_i \in X$ . Let  $L_k = \{x_{k1}, \dots, x_{kl_k}\}$  be the set of leaves of  $T_X$  at the level  $k$  and let  $n$  be a number of levels of  $T_X$ . If  $x_i \in L_k$  and  $|L_k| \geq 2$ , then  $T_Y$  can be obtained from  $T_X$  by deleting the leaf  $x_i$  from  $T_X$ . Clearly in this case  $T_Y$  has exactly one internal node at each level except the last level.

Suppose  $x_i \in L_k$  and  $|L_k| = 1$ . Taking into consideration Lemma 2.3 the space  $X$  can be uniquely presented by a sequence of balls

$$B_n \subset B_{n-1} \subset \dots \subset B_2 \subset B_1$$

where  $B_n = L_n$  and  $B_{i-1} = B_i \cup L_{i-1}$ ,  $i = n, \dots, 2$ . Let us consider the space  $(Y, d)$ . It is clear that the following relations hold

$$(3.24) \quad \overline{B}_n \subset \overline{B}_{n-1} \subset \dots \subset \overline{B}_3 \subset \overline{B}_2$$

where  $\overline{B}_n = B_n, \dots, \overline{B}_{k+1} = B_{k+1}$ ,  $\overline{B}_k = B_{k-1} \setminus \{x_i\}, \dots, \overline{B}_2 = B_1 \setminus \{x_i\}$ . Lemma 2.3 and (3.24) imply that  $T_Y$  fulfills the desirable condition.  $\square$

**Theorem 3.14.** *Let  $(X, d)$  be a finite ultrametric space with  $|X| \geq 2$ . The following conditions are equivalent.*

- (i)  $T_X$  has exactly one internal node at each level except the last level.
- (ii) For every two distinct nonsingular balls  $B_1, B_2 \in \mathbf{B}_X$  we have either  $B_1 \subset B_2$  or  $B_2 \subset B_1$ .
- (iii) The equality

$$(3.25) \quad h(T_X) + |X| = |\mathbf{B}_X|$$

holds.

- (iv) The equality

$$h(T_Y) + |Y| = |\mathbf{B}_Y|$$

holds for every nonempty  $Y \subseteq X$ .

*Proof.* The equivalence (i)  $\Leftrightarrow$  (ii) follows from the definition of the representing trees. The equivalence (i)  $\Leftrightarrow$  (iii) can be obtained from Lemma 3.12. The implication (iv)  $\Rightarrow$  (iii) is trivial and (iii)  $\Rightarrow$  (iv) follows from (i)  $\Leftrightarrow$  (ii) and Lemma 3.13.  $\square$

Using condition (i) from the last theorem it is easy to prove that every finite ultrametric space  $(X, d)$  satisfying equality (3.25) belongs to  $\mathcal{TSI}$ . See Figure 6 for example of representing tree of finite ultrametric space which satisfies Theorem 3.14.



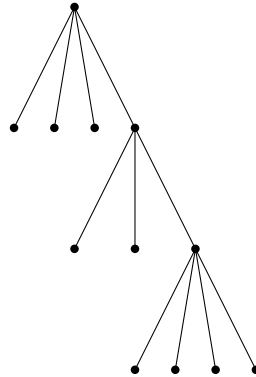


FIGURE 6. An example of a tree with exactly one internal node at each level except the last level.

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